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Reducibility Of Double Hypergeometric Series

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Abstract: *The aim of present paper, we have taken certain summation formulae due to Slater [2]; App. (III) Verma & Jain [1] and making use of making use of known identities, an attempt has been made to establish some new reducibility of double hyper geometric series into single series in original research work.*

Key Words: Generalized, geometric function, hyper geometric function, Ordinary hyper-geometric.

1. Introduction:

In this paper, we shall make use of the following well known identities:

$$\sum_{n=0}^{n=\infty} \sum_{k=0}^{n} B(n,k) = \sum_{n=0}^{n=\infty} \sum_{k=0}^{\infty} B(k,n+k), \tag{1.1}$$

An explicit representation of generalized hyper geometric functions

$$_rF_s\left[\begin{matrix} a_1, & a_2, & \dots, & a_r; & z \\ b_1, & b_2, & \dots, & b_s \end{matrix}\right] = {}_rF_s\left[\begin{matrix} (a)_r; & z \\ (b)_s \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[(a_r)]_n \ z^n}{[1]_n \ [(b_s)]_n}, \tag{1.2}$$

Valid for $|z| < 1$, provided no zeros appear in denominator. Here $a_1, a_2, a_3, \cdots, a_r$ and $b_1, b_2, b_3, \cdots, b_s$ and z are assumed to be complex number.

The shifted factorial is defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1) \dots\dots\dots\dots\dots\dots. (a+n+1); & n > 0 \end{cases} \tag{1.3}$$

In order to establish the reducibility of double hyper-geometric series into single series, we shall be need of the following known summation formulae due to (Slater [2], App.III) and Verma & Jain [1] in our analysis:

$$_3F_2\left[\begin{matrix} -n, & x, & y; & 1 \\ -n-x, & & -n-y \end{matrix}\right] = \frac{(1)_n (1+x+y)_n (1+x)m(1+y)_n}{(1+x)_n (1+y)_n m(1+x+y)_m}. \tag{1.4}$$

where m is greatest integer $\leq \frac{n}{2}$.

[Verma & Jain [1]; (2.6) P. 1024]

$$_3F_2\left[\begin{matrix} -n, & -n-2x, & y; & 1 \\ & -n-x, & 2y+1 \end{matrix}\right] = \frac{(1)_n (1+x+y)_n \ (1+x)_m (1+y)_m}{(1+x)_n \ (1+2y)_n (1+x+y)_m (1)_m}. \tag{1.5}$$

Provided that m is the greatest integer $\leq \frac{n}{2}$.

[Verma & Jain [1]; (2.13) P.1025]

$$ {}_3F_2\left[\begin{matrix}-n, & 2+n+b+2x, & 1+x; & 1\\ & 1+\frac{b}{2}+x, & (2+2x) & \end{matrix}\right] = \frac{(-)\,(1)_n\,(\frac{3}{2}+\frac{b}{2}+x)_m\,\left(1+\frac{b}{2}\right)_m}{(2+b+2x)_n(1)_m\;(\frac{3}{2}+x)_m}. \tag{1.6} $$

where m is the greatest integer $\leq \frac{n}{2}$.

[Verma & Jain [1]; (3.4) P. 1033]

$$ {}_4F_3\left[\begin{matrix}a, & 1+\frac{1}{2}a, & b, & -n; & 1\\ \frac{1}{2}a, & 1+a-b, & 1+a+n & & \end{matrix}\right] = \frac{(1+a)_n}{(1+a-b)_n}. \tag{1.7} $$

[Slater [2]; App. III (III.II)]

$$ {}_3F_2\left[\begin{matrix}a, & b, & -n; & 1\\ & 1+a-b, & 1+2b-n & \end{matrix}\right] = \frac{(a-2b)_n\,(1+\frac{1}{2}a-b)_n\,(-b)_n}{1+a-b)_n\,(\frac{1}{2}a-b)_n\,(-2b)_n}. \tag{1.8} $$

[Slater [2]; App, III (III.16)]

$$ {}_4F_3\left[\begin{matrix}a, & 1+\frac{1}{2}a, & b, & -n; & 1\\ & \frac{1}{2}a, & 1+a-b, & 1+2b-n & \end{matrix}\right] = \frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n}. \tag{1.9} $$

[Slater [2]; App. III (III.17)]

$$ {}_3F_2\left[\begin{matrix}a, & 1+\frac{1}{2}a, & -n; & 1\\ & \frac{1}{2}a, & 1+a+n & \end{matrix}\right] = (1+a)_n. \tag{1.10} $$

[Slater [2]; (2.3.4.10) p.57]

$$ {}_3F_2\left[\begin{matrix}a, & b, & -n; & 1\\ & c, & d & \end{matrix}\right] = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n} \tag{1.11} $$

Provided that c + d = a+b-n+1,

[Slater [2]; App. (III.2)]

2. Main Results:

In this section, we shall establish our main results:

2.1 Summation (1.4) can be written as:

$$ \sum_{k=0}^{n} \frac{(-n)_k\,(x)_k\,(y)_k}{k!(-n-x)_k(-n-y)_k} = \frac{(1)_n\,(1+x+y)_n\,(1+y)_m(1+y)_m}{(1+x)_n\,(1+y)_n\,(1)_n\,(1+x+y)_m}, \tag{2.1.1} $$

Multiply both side by $A_n Z^n$ and summing over n from 0 to ∞ in (2.1.1), we have

$$ \sum_{n=0}^{n=\infty}\sum_{k=0}^{n} \frac{A_n Z^n\,(-n)_k\,(-n-x)_k\,(1+y)_k}{k!(1+x)_k(1-n-y)_k} = \sum_{n=0}^{\infty} \frac{A_n Z^n\,(-)^n\,(1)_n\,(1+x+y)_n\,(1+x)_n\,(1+y)_n}{(x)_n\,(y)_n\,(1+x+y)_m\,(1)_m}, \tag{2.1.2} $$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{(1+y)_{n+k}}{(1+x)_{n+k}\,(1)_n}$ and taking $B_n = 1$ in (2.1.2), we get:

$$1^{F_1}[1+y, 1+x; z] \times 1^{F_1}[y, 1+x; -z] = \sum_{n=0}^{\infty} \frac{Z^n\,(1+x-y)_n\,(1+y)_m}{(1)_m(1+x)_n\,(1+y)_m} \quad (2.1.3)$$

Where m is the greatest integer $\leq \frac{n}{2}$.

2.2 Next, Summation (1.5) can be written as:

$$\sum_{k=0}^{n} \frac{(-n)_k\,(-n-2x)_k\,(y)_k}{k!(-n-x)_k(1+2y)_k} = \frac{(1)_n(1+x+y)_n\;(1+x)_m\;(1+y)_m}{(1+x)_n\;(1+2y)_m(1+x+y)_m\,(1)_m}, \quad (2.2.1)$$

Multiply both side by $A_n Z^n$ and summing over n from 0 to ∞ in (2.2.1), we have

$$\sum_{n=0}^{n=\infty} \sum_{k=0}^{n} \frac{A_n Z^n\,(-n)_k\,(-n-2x)_k\,(y)_k}{k!(-n-x)_k(1+2y)_k} = \sum_{n=0}^{\infty} \frac{A_n Z^n\,(1)_n\,(1+x+y)_n\,(1+x)_m\;(1+y)_m}{(1+x)_n\,(1+2y)_n\;(1+x+y)_m\,(1)_m}, \quad (2.2.2)$$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{(1+x)_{n+k}}{(1+2x)_{n+k}\,(1)_{n+k}}$ and taking $B_{n+k} = 1$ in (2.2.2), we get:

$$1^{F_1}[1+x;\;1+2y;\;z] \times 1^{F_1}[y;\;1+2y;\;-z] = \sum_{n=0}^{\infty} \frac{Z^n\,(1+x-y)_n\,(1+x)_m(1+y)_m}{(1+2x)_n\,(1+2y)_n\,(1+x+y)_m\,(1)_m} \quad (2.2.3)$$

Where m is the greatest integer $\leq \frac{n}{2}$

2.3 Further, Summation (1.6) can be written as:

$$\sum_{n=0}^{n=\infty} \frac{(-n)_k\,(2+n+b+2x)_k\,(1+x)_k}{k!(1+\frac{b}{2}+x)_k(1+2x)_k} = \frac{(-)^n\,(1)_n(\frac{3}{2}+\frac{b}{2}+k)_m\,\left(1+\frac{b}{2}\right)_m}{(2+b+2x)_n\;(\frac{3}{2}+x)_m\;(1)_m}, \quad (2.3.1)$$

Multiply both side by $A_n Z^n$ and summing over n from 0 to ∞ in (2.3.1), we have:

$$\sum_{n=0}^{\infty} \sum_{n=0}^{n=\infty} \frac{A_n Z^n\,(-n)_k\,(2+b+2x)_k\,(1+x)_k}{k!(1+\frac{b}{2}+x)_k(1+2x)_k} = \sum_{n=0}^{\infty} \frac{A_n Z^n(-)^n\,(1)_n(\frac{3}{2}+\frac{b}{2}+x)_m\,\left(1+\frac{b}{2}\right)_m}{(2+b+2x)_n\;(\frac{3}{2}+x)_m\;(1)_m}, \quad (2.3.2)$$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{(2+b+2x)_{n+k}\;B_{n+k}}{(1)_{n+k}}$ and taking $B_{n+k} = 1$ in (2.3.2), we get:

$$\sum_{k,n=0}^{n=\infty} \frac{Z^n\,(2+b+2x)_{n+2k}\,(1+x)_k(-Z)^k}{k!\;n!(1+\frac{b}{2}+x)_k(1+2x)_k} = \sum_{n=0}^{\infty} \frac{Z^n(-)^n\,(\frac{3}{2}+\frac{b}{2}+x)_m\,\left(1+\frac{b}{2}\right)_m}{(\frac{3}{2}+x)_m\;(1)_m}, \quad (2.3.3)$$

Where m is the greatest integer $\leq \frac{n}{2}$

2.4 Further, Summation (1.7) can be written as:

$$\sum_{n=0}^{n=\infty} \frac{(-1)^k\,(a)_k\,(1+\frac{a}{2})_k\,(b)_k\,(-n)_k}{k!(\frac{a}{2})_k(1+a-b)_k\,(1+a+n)_k} = \frac{(1+a)_n}{(1+a-b)_n}, \tag{2.4.1}$$

Multiply both side by $A_n Z^n$ and summing over n from 0 to ∞ in (2.4.1), we have:

$$\sum_{n=0}^{n=\infty} \sum_{k=0}^{n} \frac{A_n Z^n\ \ (-1)^k\,(a)_k\,(1+\frac{a}{2})_k\,(b)_k(-n)_k}{k!(\frac{a}{2})_k(1+a-b)_k\,(1+a+n)_k} = \sum_{n=0}^{\infty} \frac{A_n Z^n\,(1+a)_n}{(1+a-b)_n}, \tag{2.4.2}$$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{1}{(1)_{n+k}(1+a)_{n+k}}$ B_{n+k} and taking $B_{n+k} = (1+a-b)_{n+k}$ and $B_n = \alpha_n \beta_n$ in (2.4.2),

We get:

$$\sum_{k,n=0}^{n=\infty} \frac{Z^n\,(\alpha)_{n+k}\,(\beta)_{n+k}(a)_k\left(1+\frac{a}{2}\right)_k(b)_k\,(-Z)^k}{k!\,n!(\frac{a}{2})_k(1+a-b)_k\,(1+a)_{n+2k}} = 2^{F_1}[\alpha\,,\beta;1+a-b;Z], \tag{2.4.3}$$

2.5 Next, Summation (1.8) can be written as:

$$\sum_{k=0}^{k=n} \frac{(a)_k\,(b)_k\,(-n)_k}{k!(1+a-b)_k\,(1+2b-n)_k} = \frac{(a-2b)_n\,(1+\frac{1}{2}a-b)_n\,(-b)_n}{(1+a-b)_n\,(\frac{1}{2}a-b)_n(-2b)_n}, \tag{2.5.1}$$

Multiply both side by $A_n Z^n$ and summing over n from 0 to ∞ in (2.5.1), we have:

$$\sum_{n=0}^{n=\infty} \sum_{n=0}^{n=\infty} \frac{A_n Z^n\ \ (a)_k\,(b)_k\,(-n)_k}{k!(1+a-b)_k\,(1+2b-n)_k} = \sum_{n=0}^{\infty} \frac{A_n Z^n\ \ (a-2b)_n\,(1+\frac{1}{2}a-b)_n\,(-b)_n}{(1+a-b)_n\,(\frac{1}{2}a-b)_n(-2b)_n}, \tag{2.5.2}$$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{(-2b)_{n+k}}{(1)_{n+k}}$ B_{n+k} and $B_n = 1$ in (2.5.2), we get:

$$(1-z)^{2b} \times 2^{F_1\,[a,b;1+a-b;\,z]} = 3^{F_2}\left[\begin{matrix} a-2b, & 1+\frac{1}{2}a-b, & -b; & z \\ & 1+a-b & \frac{1}{2}a-b; & \end{matrix}\right], \tag{2.5.3}$$

2.6 Setting, Summation of (1.9) can be written as:

$$\sum_{n=0}^{\infty} \frac{(a)_k\,(1+\frac{1}{2}a)_k\,(b)_k\,(-n)_k}{k!(\frac{1}{2}a)_k\,(1+a-b)_k\,(1+a2b-n)_k} = \frac{(a-2b)_n\,(-b)_n}{(1+a-b)_n\,(-2b)_n}, \tag{2.6.1}$$

Multiply both side by $A_n Z^n$ and summing over n from 0 to ∞ in (2.6.1), we have:

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$$\sum_{n=0}^{n=\infty} \sum_{k=0}^{n} \frac{A_n Z^n (a)_k (1+\frac{1}{2}a)_k (b)_k (-n)_k}{k! (\frac{1}{2}a)_k (1+a-b)_k (1+a2b-n)_k} = \sum_{n=0}^{\infty} \frac{A_n Z^n (a-2b)_n (-b)_n}{(1+a-b)_n (-2b)_n}, \quad (2.6.2)$$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{(-2b)_{n+k}}{(1)_{n+k}}$ B_{n+k} and $B_{n+k} =$ 1 in (2.6.2), we get:

$$(1-z)^{2b} \times {}_3F_2 \left[\begin{matrix} a, & 1+\frac{1}{2}a, & b; \; z \\ & \frac{1}{2}a, & 1+a-b\,; \end{matrix} \right] = {}_2F_1 \, [a-2b, -b; 1+a-b; z], \quad (2.6.3)$$

2.7 Again, Setting summation (1.10) and multiply both side by $A_n Z^n$ and summing over n

from 0 to ∞ in (2.7.1), we have:

$$\sum_{n=0}^{n=\infty} \sum_{k=0}^{n} \frac{A_n Z^n (a)_k (1+\frac{1}{2}a)_k (b)_k (-n)_k}{k! (\frac{1}{2}a)_k (1+a+n)_k} = \sum_{n=0}^{\infty} A_n Z^n \, (1+a)_n, \quad (2.7.1)$$

Now, applying the identity (1.1) and replacing A_{n+k} by $\frac{1}{(1)_{n+k}(1+a)_{n+k}}$ B_{n+k} and taking $B_n = 1$ in (2.7.1), we get:

$$\sum_{k,n=0}^{\infty} \frac{z^n (a)_k (1+\frac{1}{2}a)_k (-z)^k}{k! n! (\frac{1}{2}a)_k (1+a)_k} = e^z, \quad (2.7.2)$$

Again, taking $B_n = a_n b_n$, z= -1 and applying [Slater [23; App. III (III.5)]], we get:

$$\sum_{k,n=0}^{\infty} \frac{(a)_{n+k} (b)_{n+k} (a)_k (1+\frac{1}{2}a)_k (-1)^k}{k! n! (1+a-b)_{n+k} (\frac{1}{2}a)_k (1+a)_{n+2k}} = \frac{\Gamma(1+a-b)\, \Gamma\left(1+\frac{1}{2}a\right)}{\Gamma\left(1+\frac{1}{2}a-b\right) \Gamma(1+a)}, \quad (2.7.3)$$

2.8 Further, Setting summation of (1.11) and multiply both side by $A_n Z^n$ and summing over n from 0 to ∞ in (2.8.1), we have:

$$\sum_{n=0}^{n=\infty} \sum_{k=0}^{\infty} \frac{A_n Z^n (a)_k (1+\frac{1}{2}a)_k (b)_k (c)_k (d)_k (e)_k (-n)_k}{k! (\frac{1}{2}a)_k (1+a-b)_k (1+a-c)_k (1+a-d)_k (1+a-e)_k (1+a+n)_k} =$$

$$\sum_{n=0}^{\infty} \frac{A_n Z^n (1+a)_n (1+a-b-c)_n (1+a-b-d)_n (1+a-c-d-n)_n}{(1+a-b)_n (1+a-c)_n (1+a-d)_n (1+a-b-c-d)_n}, \quad (2.8.1)$$

Now, applying the identity (1.1) and replacing A_{n+k} by $\frac{1}{(1)_{n+k}(1+a)_{n+k}}$ B_{n+k} and taking $B_n = (1+a-d)_n (1+a-b-c-d)_n$ in (2.8.1), we get:

$$\sum_{n=0}^{\infty} \frac{(1+a-d)_{n+k} (1+a-b-c-d)_{n+k} Z^n (a)_k (1+\frac{1}{2}a)_k (b)_k (c)_k (d)_k (e)_k (-Z)^{-k}}{k! n! (1+a-b)_k (1+a-c)_k (1+a-d)_k (1+a-e)_k (1+a)_{n+2k}} =$$

$${}_3F_2 \left[\begin{matrix} 1+a-b-c, & 1+a-b-d, & 1+a-c-d; & Z \\ & 1+a-b, & 1+a-c; & \end{matrix} \right], \quad (2.8.2)$$

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